A GENERALIZATION OF THE MODEL OF AN IDEAL COMPRESSIBLE FLUID

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1. We shall consider a continuous medium, in which thr free energy F and other thermodynamic potentials (internal energy U, entropy S etc.) are functions of a system of parameters

$$T, \quad \rho, \quad g^{ij}, \quad \nabla_i\rho, \quad n_i, \quad \nabla_j n_i \tag{1}$$

Here, T is the temperature, ρ is the density, g^{ij} are are the contravariant components of the metric tensor in the Lagrangian system of coordinates ξ^i at the time under consideration, $\nabla_i\rho$ are the components of the vector grad ρ in the system ξ^i, n_i are the components of some vector (for example, the vector describing the anisotropy of the medium [1]), and $\nabla_j n_i$ are the components of the gradient of this vector.

Such a medium can be considered as a generalization of the model of a compressible anisotropic fluid. Elastic media of theis kind have been introduced in [2 and 3].

We shall also assume, that for all processes in the medium under consideration

$$dq^{(e)} = TdS \qquad (dq^{(e)} \text{ is the external heat influx}) \tag{2}$$

2. It is easy to show, that if the free energy F depends on $\nabla_i\rho$ and $\nabla_j n_i$, the equation of the first law of thermodynamics for an elementary particle of the medium cannot be written in the classic form

$$dE + dU = dA^{(e)} + dq^{(e)} \tag{3}$$

Here, E is the kinetic energy and $dA^{(e)}$ is the elementary work of the external forces.

Actually, from (3), using (2) and assuming that the stress tensor p^{ij} is symmetric, we obtain the equation of the heat influx

$$dF = \frac{p^{ij}}{\rho}\, d\varepsilon_{ij} - S\, dT \tag{4}$$

If we use the relations valid in a Lagrangian system of coordinates,

$$\frac{d\rho}{dt} = -\rho g^{ij}\frac{d\varepsilon_{ij}}{dt}, \qquad \frac{dg^{kl}}{dt} = -2g^{ki}g^{lj}\frac{d\varepsilon_{ij}}{dt} \tag{5}$$

Equation (4) can be written as

$$\frac{\partial F}{\partial T} dT - \left(\rho g^{ij} \frac{\partial F}{\partial \rho} + 2g^{ki} g^{lj} \frac{\partial F}{\partial g^{kl}}\right) d\varepsilon_{ij} + \frac{\partial F}{\partial \nabla_i \rho} d\nabla_i \rho + \\ + \frac{\partial F}{\partial n_i} dn_i + \frac{\partial F}{\partial \nabla_j n_i} d\nabla_j n_i = \frac{p^{ij}}{\rho} d\varepsilon_{ij} - S\, dT \quad (6)$$

Since Equation (6) must be satisfied for all possible processes in the medium under consideration, and F and S depend only on the parameters (1) and not on the rate of change of these parameters, then either there exists some, generally speaking, nonintegrable relations between the differentials $d\nabla_i \rho$, $d\nabla_j n_i$, dT, $d\varepsilon_{ij}$ and dn_i or, if such relations are absent, the following equalities are always true

$$\frac{\partial F}{\partial \nabla_i \rho} = 0, \qquad \frac{\partial F}{\partial \nabla_j n_i} = 0 \quad (7)$$

It is easy to see, that if there exist general relations between the differentials of the parameters (1), or if (7) is satisfied, but not identically, then the system of equations for the determinations or the vector **n** in such a medium will be overspecified. This means that only displacements of some particular types can exist in the medium. However, we are considering here media in which any continuous displacements are possible. For these, it follows from (6) that Equations (7) are satisfied identically, i.e. F is independent of $\nabla_i \rho$ and $\nabla_j n_i$. This contradicts assumption (1) and proves that, for media with parametric relations (1), the first law of thermodynamics cannot be written in the form (3).

3. It is known [4], that in some cases (for example, if polarization and magnetization of the medium in the presence of an electromagnetic field are considered) we must add to the right-hand side of Equation (3) another influx of energy, different from $dA^{(e)}$ and $dq^{(e)}$. The arguments of Section 2 above show, that a change of the free energy, connected only with a change of the gradient of density, for example, cannot be brought about by mechanical work of macroscopic forces and heat influx to the particle, but be connected with an additional energy influx of a different nature. We shall denote this energy influx per unit mass by dq^{**}, and write the heat influx equation in the form (*)

$$dF = \frac{p^{ij}}{\rho} d\varepsilon_{ij} - S\, dT + dq^{**} \quad (8)$$

4. It is natural [4] to make the following assumption about dq^{**}.

1) The energy influx dq^{**} takes place through the surface of the particle, i.e.

$$\rho\, dq^{**} = \operatorname{div} \mathbf{Q}\, dt = \nabla_k Q^k dt \quad (9)$$

2) Vector **Q** becomes zero, if all parameters (1) remain constant in the particle, i.e.

$$Q^k dt = \varkappa^k dT + \Lambda^{k\cdot\cdot}_{\cdot ij} dg^{ij} + M^{ki} d\nabla_i \rho + N^{ki} dn_i + P^{kji} d\nabla_j n_i \quad (10)$$

Here we did not include a term of the type $\Lambda^k d\rho$, since, although ρ and g^{ij} are independent parameters, their differentials are related by a general equation, namely $2d\rho = \rho g_{ij} dg^{ij}$.

Using (9), (10) and (5), we can rewrite the heat influx equation

$$\frac{\partial F}{\partial T} dT - \left(\rho \frac{\partial F}{\partial \rho} g^{ij} + 2g^{ki} g^{lj} \frac{\partial F}{\partial g^{kl}} + \nabla_k \rho \frac{\partial F}{\partial \nabla_k \rho} g^{ij}\right) d\varepsilon_{ij} - \quad (11) \\ - \rho \frac{\partial F}{\partial \nabla_k \rho} g^{ij} \nabla_k d\varepsilon_{ij} + \frac{\partial F}{\partial n_i} dn_i + \frac{\partial F}{\partial \nabla_j n_i} d\nabla_j n_i = - S\, dT + \frac{p^{ij}}{\rho} d\varepsilon_{ij} + \\ + \frac{1}{\rho} \nabla_k \varkappa^k dT + \frac{1}{\rho} \varkappa^k \nabla_k dT - \frac{2}{\rho} \nabla_k \Lambda^{kij} d\varepsilon_{ij} - \frac{2}{\rho} \Lambda^{kij} \nabla_k d\varepsilon_{ij} +$$

*) The necessity of introducing dq^{**} was fully discussed in Sedov's paper at the 11th International Congress for Applied Mechanics, in Munich, 1964.

$$+\frac{1}{\rho}\ \nabla_k M^{ki}\ d\nabla_i\rho + \frac{1}{\rho} M^{ki}\nabla_k d\nabla_i\rho + \frac{1}{\rho}\ \nabla_k N^{ki} dn_i + \frac{1}{\rho} N^{ki}\nabla_k dn_i + \frac{1}{\rho}\ \nabla_k P^{kji}\ d\nabla_j n_i + \frac{1}{\rho} P^{kji}\nabla_k d\nabla_j n_i \qquad \text{(11) cont.}$$

Here, the following equation was also used:

$$d\nabla_i\rho = -\nabla_i\rho g^{kj} d\varepsilon_{kj} - \rho g^{kj}\nabla_i d\varepsilon_{kj} \qquad (12)$$

From (11) we can obtain all the defining relations for our medium, with the assumption that $\varkappa^k$, Λ^{kji}, M^{ki}, N^{ki} and p^{kji} are independent of the rate of change of the parameters (1), and between the differentials, appearing in (11), there are no general relations for the medium under consideration.

For the determination of the defining equations we shall also use the following relations where ∇_i is the covariant differentiation on ξ^i and d is the differential on time with constant ξ^i

$$d\nabla_k\rho = \nabla_k d\rho$$

$$d\nabla_k\varepsilon_{ij} = \nabla_k d\varepsilon_{ij} - (\varepsilon_j{}^s L_{kis}^{pqr} + \varepsilon_i{}^s L_{kjs}^{pqr})\nabla_p d\varepsilon_{qr}$$

$$d\nabla_k\nabla_i\rho = \nabla_k d\nabla_i\rho] - \nabla^s\rho L_{kis}^{pqr}\nabla_p d\varepsilon_{qr}$$

$$d\nabla_k n_i = \nabla_k dn_i - n^s L_{kis}^{pqr}\nabla_p d\varepsilon_{qr}$$

Here

$$2L_{ijk}^{pqr} = \delta_i{}^p(\delta_j{}^q\delta_k{}^r + \delta_k{}^q\delta_j{}^r) + \delta_j{}^p(\delta_k{}^q\delta_i{}^r + \delta_i{}^q\delta_k{}^r) - \delta_k{}^p(\delta_i{}^q\delta_j{}^r + \delta_j{}^q\delta_i{}^r) \qquad (13)$$

($\delta_k{}^p$ are Kronecker symbols)

Equations (13) become obvious, if we consider that for time differentials the Christoffel symbols Γ_{ij}^k in the Lagrangian system of coordinates are

$$d\Gamma_{ij}^k = \nabla_i\nabla_j v^k\, dt = g^{ks}\ (\nabla_i d\varepsilon_{js} + \nabla_j\ d\varepsilon_{is} - \nabla_s d\varepsilon_{ij}) = L_{ijs}^{pqr} g^{ks}\nabla_p d\varepsilon_{qr}$$

With the above assumptions, the requirement that the heat influx Equation (11) should be satisfied for all processes in the medium, leads to the defining equations

$$M^{ki} = 0, \qquad P^{kji} = 0, \qquad \varkappa^k = 0 \qquad (14)$$

$$S = -\frac{\partial F}{\partial T} \qquad (15)$$

$$N^{ji} = \rho\frac{\partial F}{\partial\nabla_j n_i} \qquad \frac{\partial F}{\partial n_i} = \frac{1}{\rho}\ \nabla_k N^{ki} \qquad (16)$$

$$\frac{2}{\rho}\Lambda^{kij} = \rho\frac{\partial F}{\partial\nabla_k\rho} g^{ij} + \frac{\partial F}{\partial\nabla_p n_q} n^s L_{pqs}^{kij} \qquad (17)$$

$$\frac{p^{ij}}{\rho} = -\left(\rho\frac{\partial F}{\partial\rho} + \nabla_k\rho\frac{\partial F}{\partial\nabla_k\rho}\right)g^{ij} - 2g^{ki}g^{lj}\frac{\partial F}{\partial g^{kl}} + \frac{2}{\rho}\ \nabla_k\Lambda^{kij} \qquad (18)$$

If the free energy is known as a function of its parameters, Equations (14), (16) and (17) can be used to calculate the energy influx dq^{**}

$$dq^{**} = \frac{1}{\rho}\ \nabla_k\left(\rho\frac{\partial F}{\partial\nabla_k\rho}\right)d\rho + \frac{\partial F}{\partial\nabla_k\rho}\ d\nabla_k\rho + \frac{\partial F}{\partial n_i} dn_i + \frac{\partial F}{\partial\nabla_k n_i}\ d\nabla_k n_i - \frac{1}{\rho}\ \nabla_k\left(\rho\frac{\partial F}{\partial\nabla_p n_q} n^s L_{pqs}^{kij}\right)d\varepsilon_{ij} \qquad (19)$$

We can see from (19) that, in particular, in the deformation of media in which F does not depend on $\nabla_i\rho$, but depends on $\nabla_j n_i$, there will, in general, be a nonzero dq^{**} even in processes with $dn_i = 0$ and $d\nabla_k n_i = 0$.

Equations (17) and (18) show that the stress tensor in such a medium is not spherical, which is also true for conditions of equilibrium. Therefore, a medium in which the free energy at a given time depends not only on the density, but also on the gradient of the density, is not a fluid in the usual sense.

We can also note that the dependence of the free energy on the gradients of density and vector **n**, leads to the dependence of the stresses, in the general case, on the gradients of all the parameters (1), in particular on the second space derivatives of ρ and **n**. This dependence on the second derivatives is linear. The investigation of media (*), in which the free energy depends on the time derivatives of the density [5], also shows that the stresses depend linearly, to a large extend, on the time derivatives of ρ. In each case, this conclusion is significantly connected with the assumption that dq^{**} is the energy influx through the surface (or is absent), and also with the reversibility condition (2).

5. To exhibit the characteristic peculiarities of the mechanical behavior of media, in which the energy depemds on the gradient of density, we shall investigate the following simple model. Let F depend only on T, ρ, g^{ij} and $\nabla_i\rho$, i.e.

$$F = F(T, \rho, \mu), \qquad \mu = g^{ij}\nabla_i\rho \; \nabla_j\rho \tag{20}$$

In addition assume that

$$F = F_1 + \frac{k^2}{2\rho^2}\mu \tag{21}$$

where $F_1 = F_1(\rho, T)$ is the free energy of a mass unit of ideal gas and k^2 is a constant. The dimensions of k^2 are the dimensions of the product of F with the square of some length l. Thus, in a number of parameters, describing the medium, we get a linear behavior [6]. In the statement of specific problems there is usually involved some length L, describing the objects relevant to the problem. Apparently, the improvement of the ideal gas model, given by Equation (21), can become effective in problems where $L^2 \sim l^2$ or $L^2 \ll l^2$.

According to (15), the expression for the entropy of the medium will coincide with the expression for the entropy of the ideal gas, and, in particular, the adiabatic condition (coinciding here with the condition S = const) will be $T = c\rho^{\gamma-1}$

We shall also write the equations for the stress tensor componenets (coinciding with Clapeyron's equation for $k = 0$)

$$p^{ij} = -\left(R\rho T + \frac{k^2}{\rho}\mu - k^2\nabla_\alpha\nabla^\alpha\rho\right)g^{ij} - \frac{k^2}{\rho}\nabla^i\nabla^j\rho \tag{22}$$

and the closed system of equations, describing the adiabatic motions of the medium with no body forces in an Eulerian system of coordinates

$$\frac{d\rho}{dt} + \rho\nabla_k v^k = 0 \tag{23}$$

$$\rho\frac{dv^i}{dt} = -A\rho^{\gamma-1}\nabla^i\rho + \frac{2k^2}{\rho^2}\nabla^i\rho\nabla_j\rho\nabla^j\rho - \frac{3k^2}{\rho}\nabla_j\nabla^i\rho\nabla^j\rho - \frac{k^2}{\rho}\nabla^i\rho\nabla_j\nabla^j\rho + k^2\nabla^i\nabla_j\nabla^j\rho$$

$$\left(A = \frac{\gamma R T_0}{\rho_0^{\gamma-1}} = \text{const}\right) \tag{24}$$

Consider small ($\rho = \rho_0 + \rho'$, ρ' and v^i and their derivatives being small) unidimensional motions with plane waves. Obviously, with such an approximation we can only obtain longitudinal waves in the medium. This is connected with the fact, that to a first order approximation the stress tensor is here spherical.

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*) Such media were investigated in Eglit's dissertation, MGU, 1962.

For longitudinal waves, travelling along the x-axis, we get Equation

$$\frac{\partial^2 \rho'}{\partial t^2} = a_0^2 \frac{\partial^2 \rho}{\partial x^2} - k^2 \frac{\partial^4 \rho'}{\partial x^4} \qquad (a_0^2 = \gamma R T_0) \tag{25}$$

Here a_0 is the sound velocity in the ideal gas.

Equation (25) has a solution of the form $\exp[i(\alpha x - \omega t)]$, and the wavelength is connected with the frequency of the dispersion equation

$$\omega = \pm \alpha \sqrt{a_0^2 + k^2\alpha^2} \tag{26}$$

Hence, with the consideration of the dependence of the free energy on the gradient of density we find a sound dispersion in the medium. This effect exists for short waves ($k^2\alpha^2 \sim a_0^2$) and not for long ones ($k^2\alpha^2 \ll a_0^2$). By measuring the dispersion of short waves we can find the magnitude of k^2 for the medium under consideration.

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